Homotopy in the Category of Graphs
University of Montana: Department of Mathematical Sciences Colloquium

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MSUBILLINGS
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Outline of the Presentation

1. Homotopy and Graphs
2. The Exponential Object
3. Retracts
4. The Homotopy Category
Throughout this talk, the “Category of graphs” is the category where:

- **Objects** are Graphs with:
  - Finite number of vertices.
  - Loops are allowed.
  - At most 1 edge between vertices.

- **Morphisms** map vertices to vertices, edges to edges, and preserve adjacency.

This would be “Finite Single Strict Graphs” in the framework of McRae, Plessas and Rafferty.
We understand homotopy to be a continuous transformation of maps, which doesn’t make sense in the discrete world of graphs.
Transition from Spaces to Graphs

How do we move from the continuous world Top to the discrete world Graphs?

In Top, continuous functions preserve Convergence.

In Graphs we’re interested in functions which preserve Adjacency.

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Definition

Given graphs $G, H$, a **Graph Homomorphism** $f : G \to H$ is a map $f : V(G) \to V(H)$ such that if $vw \in E(G)$, then $f(v)f(w) \in E(H)$. 
Transition from Spaces to Graphs: Products

What does $\times$ mean as we move from Top to Graphs?

In Top, the product is the cartesian product with the product topology.

In Graphs the vertices are the cartesian product, and there is an edge between $(v_1, w_1) \sim (v_2, w_2)$ iff $v_1 \sim v_2$ and $w_1 \sim w_2$.

Each product has an appropriate projective morphism to the factor objects.

Let $I_n$ denote the graph:
In light of this, how does Homotopy change?

In Top, two functions being homotopic means a continuous “transformation” from one map to another:

In Graphs we replace continuous function with graph homomorphism, using our new product:
**Exponential Object**

**Definition**

Let $\mathcal{C}$ be a category with binary products and let $Z$ and $Y$ be objects in $\mathcal{C}$. An object $Z^Y$ together with a morphism $\text{eval} : Z \times Y \to Z$ is an exponential object if for any object $X$ and morphism $g : X \times Y \to Z$ there is a unique morphism $\lambda_g : X \to Z^Y$ such that the following diagram commutes:

\[
\begin{array}{ccc}
X & \rightarrow & X \times Y \\
\downarrow \exists! \lambda_g & \nearrow \exists! \lambda_g \times \text{id}_Y & \\
Z^Y & \rightarrow & Z^Y \times Y \\
\downarrow \text{eval} & & \downarrow \text{eval} \\
& & Z \\
\end{array}
\]

In other words $\text{Hom}(X, Z^Y) \cong \text{Hom}(X \times Y, Z)$ in a natural way.
**Exponential Object in Graphs**

**Theorem (Dochtermann ’09)**

For graphs $G, H$, let $H^G$ denote the graph where $V(H^G) = \{ f \mid f : G \to H \}$ and $fg \in E(H^G)$ iff whenever $vw \in E(G)$ then $f(v)g(w) \in E(H)$. Then $H^G$ is the exponential object of the Category of Graphs.
The Exponential Object in Graphs

Theorem (Dochtermann ’09)

For graphs $G, H$, let $H^G$ denote the graph where $V(H^G) = \{ f \mid f : G \to H \}$ and $fg \in E(H^G)$ iff whenever $vw \in E(G)$ then $f(v)g(w) \in E(H)$. Then $H^G$ is the exponential object of the Category of Graphs.
Homotopy in Graphs

**Definition**

Graph homomorphisms \( f, g : G \to H \) are **homotopic** if there is an injection \( \iota \) from \( I_n \to H^G \) such that \( \iota(0) = f, \iota(n) = g \).
Homotopy in the Exponential Object

Definition (Dochtermann ’09, paraphrased)

Graph homomorphisms $f, g : G \to H$ are homotopic if there is a path from $f$ to $g$ in the subgraph of $H^G$ induced by morphisms.
**Definition**

Let \( f, g : G \to H \). We say that \( f \) and \( g \) are a **spider pair** if there is a single vertex of \( G \), say \( x \), such that \( f(y) = g(y) \) for all \( x \neq y \). If \( x \) is unlooped there are no additional conditions, but if \( x \_x \) in \( G \), then we ask that \( f(x) \_g(x) \).

**Lemma**

If \( f \) and \( g \) are a spider pair, then \( f \_g \) in \( H^G \).
Let $G$ and $H$ be the following graphs:

$$G = \begin{array}{c}
0 \\
\hline
1
\end{array}, \quad H = \begin{array}{c}
a \\
\hline
b \\
\hline
c
\end{array}.$$

Let $f : G \to H$, $f(0) = a, f(1) = b$, and $g : G \to H$, $g(0) = a, g(1) = a$. 

\[\text{Img}(f)\]  
\[\text{Img}(g)\]
Morphisms $f, g$ in $H^G$
More Spiders

The identity map and $\rho$ are a spider pair.
More Spiders

Morphisms $f, g$, are a spider pair.
Spider Lemma

Proposition (Spider Lemma! AKA one leg at a time)

If \( f \_ g \) in \( H^G \), then there is a finite sequence of morphisms \( f = f_0, f_1, f_2, \ldots, f_n = g \) such that each successive pair \( f_k, f_{k+1} \) is a spider pair.

\[
f_k(v_i) = \begin{cases} f(v_i) & \text{for } i \leq n - k \\ g(v_i) & \text{for } i > n - k \end{cases}
\]

Proof.

- \( f_k \) is a morphism: Given \( v, w \in V(G) \) where \( v \_ w \), \( f(v) \_ f(w), g(v) \_ v(w) \) since \( f, g \) are morphisms, \( f(v) \_ g(w), f(w) \_ g(v) \) since \( f \_ g \) in \( H^G \).

- If \( v_{k+1} \) is looped, then since \( f \_ g \), we know that \( f(v_{k+1}) \_ g(v_{k+1}) \), so \( f_k(v_{k+1}) \_ f_{k+1}(v_{k+1}) \).
Corollary

Whenever \( f \simeq g \), there is a finite sequence of spider pairs connecting \( f \) and \( g \).

Definition (Hell & Nestril)

Hom\((G, H)\) is a graph where the vertex set is taken to be the collection of morphisms from \( G \) to \( H \), and two morphisms are adjacent if they differ on at most one vertex.

Corollary

Hom\((G, H)\) as defined in Hell & Nestril is a subgraph of \( H^G \).
Decomposition into Spiders

Let $G = C_4$ and $H = P_2$ with vertices labeled as below.

Let $G = C_4$ and $H = P_2$ with vertices labeled as below.

With morphisms $f, g$: $f(1)$, $f(0)$, $f(2)$, $f(3)$, $g(3)$, $g(0)$, $g(2)$, $g(1)$

Img($f$) Img($g$)

Note, $f$, $g$ are not a spider pair.
Decomposition into Spiders

$f, g$ homotopic

Diagram:

```
        baba
   ________________________
  |                         |
  |                         |
  |                         |
  |                         |
  |     babc               |
  |                         |
  |                         |
  |                         |
  |     f                 |
  |                         |
  |                         |
  |                         |
  |                         |
  |     gb                 |
  |                         |
  |                         |
  |                         |
  |_____________________
        bcba
```
Decomposition into Spiders

We introduce the morphism $h$.

\[ \text{Img}(f) \quad \text{Img}(h) \quad \text{Img}(g) \]
Folds

Definition

If $G$ is a graph, we say that a morphism $f : G \to G$ is a fold if $f$ and the identity map are a spider pair.

$u \Rightarrow v \quad \rho \quad u = \rho(v)$

Proposition

If $f$ is a fold, then $f : G \to \text{im}(f)$ is a homotopy equivalence.

Koslov proves this in ’06 using simplicial complexes. We have a proof internal to graphs.
Proposition

Suppose that $f : V(G) \to V(G)$ is a set map of vertices such that $f$ is the identity on all vertices except $w$. Explicitly, there exists a vertex $w \in V(G)$, and $f(x) = x$ for all $x \neq w$. Let $v = f(w)$. Then $f$ is a fold if and only if $N(w) \subseteq N(v)$.

In the literature, a fold is sometimes called a **dismantling**.

Definition

A graph is called **stiff** if for any pair of vertices $v, w$, it follows that $N(w) \nsubseteq N(v)$ and $N(v) \nsubseteq N(w)$.
**Lemma**

*If* $G$ *is stiff, then* $\text{id}_G$ *is not homotopic to any other endomorphism.*

**Proof.**

Suppose that $f \neq \text{id}_G$. Let $v$ be any vertex of $G$, and let $x \in N(v)$. Then $f(v) \not\sim \text{id}_G(x)$, i.e. $f(v) \sim x$ so $x \in N(f(v))$. So $N(v) \subseteq N(f(v))$. Since $G$ is stiff, we conclude that $v = f(v)$ and so $f = \text{id}$.

**Definition**

A graph $G$ is **homotopy minimal** if it is not homotopy equivalent to any proper subgraph of itself.

**Theorem**

$M$ is homotopy minimal if and only if $M$ is stiff.
Lemma

If $f : G \to G$ such that $f \_ id_G$ then $G$ is homotopy equivalent to $im(f)$.

Proof.

- $\iota : im(f) \to G$. Clear $\iota f = f$ homotopic to $id_G$.
- Suppose that $v \_ w \in E(H)$.
- $\exists v' \_ w' \in E(G)$, where $f(v') = v, f(w') = w$.
- $f \_ id_G \Rightarrow v' \_ f(w') \in E(G)$.
- $f(v') \_ f(f(w')) \in E(H)$.
- So $v \_ f \iota(w)$ whenever $v \_ w$, and so $id_H \_ f \iota$. 

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Homotopy in the Category of Graphs
Theorem

Given any finite graph $G$, there is a unique (up to isomorphism) homotopy minimal graph $M$ such that $G$ is homotopy equivalent to $M$.

Proof.

- Induction.
- Let $M, M'$ both be homotopy minimal graphs equivalent to $G$. Then $\exists f : M \to M', g : M' \to M$ are homotopy equivalences. Then $gf = \text{id}_M, fg = \text{id}_{M'}$.

Definition (Shoutout to Jeffery Johnson!)

Let $\mathcal{P}(G)$ denote the homotopy minimal graph of $G$.

Hell & Nestril have an equivalent result, that every graph folds down to a unique (up to iso) stiff graph.
Folds to isomorphic copies
What are pleats?

Some examples of pleats:

- Cores.
- Even cycles of length $> 4$.
- Other stuff.
Observe $\mathcal{P}(X \coprod Y) \cong \mathcal{P}(X) \coprod \mathcal{P}(Y)$. If $X$, $Y$ have no isolated vertices:

**Proposition**

Let $X$, $Y$ be graphs, then $\mathcal{P}(X) \times \mathcal{P}(Y)$ is stiff.

**Lemma**

Let $w \in V(X)$ such that there is a fold $\rho : X \to X \setminus \{w\}$. Then $X \times Y$ is homotopy equivalent to $X \setminus \{w\} \times Y$.

**Theorem**

Let $X$, $Y$ be graphs, then $\mathcal{P}(X \times Y) \cong \mathcal{P}(X) \times \mathcal{P}(Y)$. 
A 2–Category

Definition (Riehl ’16, CTIC)

A 2–category consists of:

- A collection of Objects (or 0–cells).
- A collection of Morphisms (or 1–cells, 1–morphisms) between pairs of objects.

\[
\begin{array}{c}
X \\
\downarrow f \\
Y
\end{array}
\]

- A collection of 2–morphisms (or 2–cells) between parallel pairs of morphisms.

\[
\begin{array}{c}
X \\
\downarrow \alpha \\
Y
\end{array}
\]

Such that:

- The objects and morphisms form a category.
- Given objects \(X, Y\), the morphisms \(\text{Hom}(X, Y)\) and the 2–morphisms between them form a category.
That the composition of 2–cells defined as:

\[
\begin{align*}
X & \xrightarrow{f} Y \\
Y & \xrightarrow{g} Z \\
X & \xrightarrow{g' \circ f'} Z
\end{align*}
\]

\[
\begin{align*}
X & \xrightarrow{f'} Y \\
Y & \xrightarrow{g'} Z \\
X & \xrightarrow{g' \circ f'} Z
\end{align*}
\]

is unital and associative.

That the interchange law holds:

\[
(\beta' \circ \beta) \circ (\alpha' \alpha) = (\beta \circ \alpha)(\beta' \circ \alpha')
\]
Objects are graphs.
Morphisms are graph homomorphisms.
2–morphisms are homotopy classes of homotopies.

Our objects and morphisms already form a category!
Homotopy and Graphs

The Exponential Object

Retracts

The Homotopy Category

Concactanation of Homotopies

Definition

Given a path $\alpha : I_{n_1}^\ell \to G$ from $x$ to $y$, and a path $\beta : I_{n_2}^\ell \to G$ from $y$ to $z$, we define the **concatenation of paths** $\alpha \ast \beta : I_{n_1+n_2}^\ell \to G$ by

$$(\alpha \ast \beta)(i) = \begin{cases} 
\alpha(i) & \text{if } i \leq n_1 \\
\beta(i-n_1) & \text{if } n_1 < i \leq n_1 + n_2
\end{cases}$$

Since we are assuming that $\alpha(v_{n_1}) = y = \beta(v_1)$, this defines a graph morphism $I_{n_1+n_2}^\ell \to G$ from $x$ to $z$.

Definition (Concatenation of Homotopies)

Given $\Lambda_1 : f \simeq g$ and $\Lambda_2 : g \simeq h$, we define $\Lambda_1 \ast \Lambda_2 : f \simeq h$ using the concatenation of paths in $G^H$. 
Let $X = C_4$, $Y = P_3$. Define $f_0, f_1, f_2, f_3$ as follows:

So we have 1-homotopies: $\alpha_1, \alpha_2, \alpha_3$ where $\alpha_1 : f_0 \Rightarrow f_1$, $\alpha_2 : f_1 \Rightarrow f_2$ and $\alpha_3 : f_2 \Rightarrow f_3$. 
We can see that \((\alpha_3 \alpha_2 \alpha_1)\) and \(\alpha_3 (\alpha_2 \alpha_1)\) produce the same homotopy.
Hom\((G, H)\) is a category

- We define \([\alpha'][\alpha] = [\alpha * \alpha']\) (this is well defined).
- The length-0 identity homotopy serves as the identity morphism.
- Concatenation of paths is associative.
Composition of Homotopies

Definition (Composition of Homotopies)

Let \( f, f' : X \to Y \) and \( g, g' : Y \to Z \) be graph homomorphisms such that there is a \( n_1 \)-homotopy \( \alpha \) from \( f \) to \( f' \) and a \( n_2 \)-homotopy \( \beta \) from \( g \) to \( g' \). Then define \( \beta \circ \alpha : I_{n_1+n_2} \to Z^X \) such that:

\[
(\beta \circ \alpha)(i) = \begin{cases} 
\beta(0)\alpha(i) & i \leq n_1, \\
\beta(i - n_1)\alpha(n_1) & i > n_1.
\end{cases}
\]
Example of Homotopy Composition

Let \( X = C_4, Y = Z = P_3 \). Consider the following morphisms.

\[
\begin{align*}
\alpha &: f \Rightarrow f', \\
\beta &: g \Rightarrow g',
\end{align*}
\]
that is \( \alpha(0) = f, \alpha(1) = f', \beta(0) = g, \beta(1) = g' \).
Then $\beta \circ_0 \alpha$ is a 2-homotopy $\beta \circ_0 \alpha : I^2 \to Z^X$ such that

$$(\beta \circ_0 \alpha)(0) = g \circ f, \ (\beta \circ_0 \alpha)(1) = g \circ f', \ (\beta \circ_0 \alpha)(2) = g' \circ f'. $$
Composition of 2–cells is unital and associative

- We define \([\beta] \circ [\alpha] = [\beta \circ \alpha]\) (this is well defined).
- The length-0 identity homotopy serves as the unit.
- Composition of morphisms is associative.
Interchange Law

Notice that the following are a spider pair in $Z^X$:

So consider:

\[
\begin{align*}
\alpha &\circ (\alpha' \beta) \\
(\beta' \circ \alpha')(\beta \circ \alpha)
\end{align*}
\]
To Summarize

Theorem

We can define a 2-category $Gph$ as follows:

- Objects (0-cells) are given by objects of $Gph$, the finite undirected graphs.
- Arrows (1-cells) are given by the arrows of $Gph$, the graph morphisms.
- Given $f, f' : G \rightarrow H$, a 2-cell is a homotopy class $[\alpha]$ of homotopies $\alpha : I_n^\ell \rightarrow H^G$.
- Vertical composition is defined using concatenation $[\alpha] \ast [\alpha'] = [\alpha \ast \alpha']$.
- Horizontal compositon is defined using composition $[\beta] \circ [\alpha] = [\beta \circ \alpha]$.

Definition

We define the homotopy category $HoGph$ by modding the 2-category $Gph$ out by the 2-cells.

- Objects are graphs.
- Morphisms are homotopy classes of graph homomorphisms $[\alpha]$.

This also defines a natural projection functor $\Psi : Gph \rightarrow HoGph$. 
The Homotopy Category

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**Theorem**

\( \text{HoGph} \) deserves the name "The Homotopy Category".

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**Ok maybe a bit more formally...**

\( \text{HoGph} \) is the homotopy category for \( \text{Gph} \). Explicitly, there is a functor \( F : \text{Gph} \rightarrow \mathcal{X} \) such that \( F \) takes homotopy equivalences to isomorphisms, then there is a functor \( F' : \text{HoGph} \rightarrow \mathcal{X} \) such that \( F'\Psi = F \).

\[
\begin{align*}
\text{Gph} & \xrightarrow{\Psi} \text{HoGph} \\
& \downarrow F \\
\text{HoGph} & \xrightarrow{\exists! F'} \mathcal{X}
\end{align*}
\]
Proof of Theorem

The only sensible choice of $F'$:

- $F'(X) = F(X)$ for any $X \in \text{Obj}(Gph)$.
- $F'([\varphi]) = F(\varphi)$ for any $[\varphi] \in \text{Hom}(\text{HoGph})$.

First is obviously well-defined.
Second, not so much.
Proof of Well-Definedness

Proof.
Let \( f_1, f_2 : X \rightarrow Y \) such that \( f_1, f_2 \) are a spider pair. Need to show \( F(f_1) = F(f_2) \). Let \( v \) be the vertex they differ on.

- Define \( \hat{X} \):

\[
\begin{align*}
V(\hat{X}) &= V(X) \cup \{v^*\}. \\
E(\hat{X}) &= \begin{cases} 
\text{when } w_1w_2 \in E(X), \\
\text{when } v_w \in E(X), \\
\text{when } v_v \in E(X)
\end{cases}
\end{align*}
\]

- Let \( \iota_i : X \rightarrow \hat{X} \) where \( \iota_1(v) = v, \iota_2(v) = v^* \).

- Define \( \hat{f} : \hat{X} \rightarrow Y \) by

\[
\hat{f}(w) = \begin{cases}
    f_1(w) & w \in V(X) \\
    f_2(v) & w = v^*
\end{cases}
\]

- Notice \( f_i = \hat{f}_{\iota_i} \).
Proof of Well-Definedness

Proof.

- Let $\rho : \hat{X} \to X$ where $\rho(v^*) = v$. Notice that $\rho$ is a fold, each $\iota_i$ is a homotopy equivalence, $\rho \iota_i = \text{id}_X$.
- Notice $F(\rho), F(\iota_i)$ are isomorphisms in $\mathcal{X}$, and since $F(\rho \iota_1) = F(\text{id}_X) = F(\rho \iota_2)$, we have that $F(\rho)F(\iota_1) = F(\rho)F(\iota_2)$ and $F(\iota_1) = F(\iota_2)$.
- So:

$$
\begin{align*}
F(f_1) & = F(\hat{f} \iota_1) \\
& = F(\hat{f})F(\iota_1) \\
& = F(\hat{f})F(\iota_2) \\
& = F(\hat{f} \iota_2) \\
& = F(f_2).
\end{align*}
$$

Thus, given $f_1, f_2 \in [f]$, we have that $F'([f_1]) = F(f_1) = F(f_2) = F'([f_2])$ and $F'$ is well defined.
We also define a groupoid $\Pi(G)$ of a graph $G$.

Come see my talk in Portland on Saturday!
Future Work

For Us:

- Identify “cool” homotopy invariants. (We have some but not “cool” enough). In particular, find a cool functor to HoGph.
- Study the factorization of homotopy equivalences as folds and un-folds.
- Further exploration of the Fundamental Groupoid.

For Students:

- Play in Sage.
- Automorphisms homotopic to the identity form a normal subgroup.
I’d like to thank the University of Montana Mathematics Department for welcoming me back to give this talk, and giving me the training and education that I needed to do this as a career. I would like to thank Dr. Laura Scull for the fun and fruitful collaboration!! I’d like to thank Dr. Demitri Plessas for being a sounding board for my ideas even though he’s abandoned us to make money.

Any Questions?