1. Let $G$ be a group and recall that the non-negative powers of an element $x$ in $G$ are defined as follows: $x^0 = e$, $x^1 = x$, and $x^{n+1} = x^n x$ for $n > 0$. Now, let $x$ and $y$ be elements in $G$ and, using induction, show that $(x y x^{-1})^n = x y^n x^{-1}$ holds for $n > 0$.

2. What property of the integers states that any nonempty set, $S$, of positive integers has a least element (i.e. an $m \in S$ such that $m \leq x$ for all $x \in S$)?

3. The following set of congruence classes mod 7 forms a group under multiplication: $\{[1], [2], [3], [4], [5], [6]\}$. Identify the following:


4. Solve the following system in $\mathbb{Z}_7$: $[2][x] + [1][y] = [6]$ & $[4][x] + [5][y] = [6]$
5. Let $\phi : G \rightarrow G'$, a mapping from one group $G$ to another group $G'$, be a homomorphism. Let $e_G$ be the identity in $G$ and $e_{G'}$ the identity in $G'$. Consider the following statements:

1. $\phi$ takes the identity of $G$ to the identity of $G'$, i.e. $\phi(e_G) = e_{G'}$.
2. For $x \in G$, $\phi(x^{-1}) = (\phi(x))^{-1}$, i.e. $\phi$ maps inverses to inverses.
3. The kernel of $\phi$ is a normal subgroup of $G$.
4. The image of $\phi$ is a normal subgroup of $G'$.

Then

(a) statement 4 is false, but the others are true.
(b) statements 1 & 2 are true, but not 3 & 4.
(c) each of these statements is true.
(d) only statements 1 & 4 are true.

6. In proving the uniqueness of the group identity, which of the following properties is needed?

(a) the commutative property
(b) the distributive property
(c) the associative property
(d) none of these

7. In showing the uniqueness of inverses in a group, which of the following properties is needed?

(a) the commutative property
(b) the distributive property
(c) the associative property
(d) none of these
8. Consider the relation $R$ defined on the set $\mathbb{Z}$ of integers:

$xRy$ if and only if 7 divides $5x - 12y$

(a) Show that $R$ is an equivalence relation:

$R$ is reflexive:

$R$ is symmetric:

$R$ is transitive:

(b) Describe the distinct equivalence classes of $R$:

9. A subset $S$ of the ring $R$ is a subring of $R$ if and only if the following conditions hold:

(a) $S$ is nonempty, and
(b) For any $x, y \in S$, $x - y$ & $x \cdot y \in S$

True False

10. In a multiplicative group $G$, given that we’ve defined $y^n$ for $y \in G$ and $n$ a positive integer, how should we define $x^k$ for $x \in G$ if $k$ is a negative integer?

11. If every proper subgroup of a group $G$ is cyclic, then $G$ itself must be cyclic.

True False

12. If $K$ is a subgroup of $H$ and $H$ is a subgroup of $G$, then $K$ is a subgroup of $G$.

True False
13. Show the mapping \( \phi : \mathbb{C}\{0\} \to H \) defined by \( \phi(a + bi) = \begin{bmatrix} a & -b \\ b & a \end{bmatrix} \) is an isomorphism from the multiplicative group of non-zero complex numbers \( \mathbb{C}\{0\} \) to the multiplicative group \( H = \{ \begin{bmatrix} a & -b \\ b & a \end{bmatrix} : a \& b \text{ real with } a^2 + b^2 \neq 0 \} \).

Fill in the blanks:

14. If \( \phi \) is an epimorphism from the group \( G \) to the group \( G' \) with \( K = \ker \phi \), then \( G' \) is isomorphic to \( G/__________ \).

15. Every field is an integral domain. When is an integral domain a field?
   If the integral domain is __________.