Second proof: Let $k = \dim T(H)$. If $k = 0$, then $k < \dim H$. Otherwise, $T(H)$ has a basis which can be written in the form $T(\mathbf{v}_1), \ldots, T(\mathbf{v}_k)$ for some vectors $\mathbf{v}_1, \ldots, \mathbf{v}_k$ in H. Since $\{T(\mathbf{v}_1), \ldots, T(\mathbf{v}_k)\}$ is linearly independent, so is $\{\mathbf{v}_1, \ldots, \mathbf{v}_k\}$, by Exercise 31 in Section 4.3. Since $\mathbf{v}_1, \ldots, \mathbf{v}_k$ are in H, the dimension of H must be at least k.

Hint for Exercise 32: Use an exercise in Section 4.3.

Study Tip: The next section is quite important. Do your best to get caught up now. Otherwise, you may have difficulty relating the various concepts and facts about matrices that will be reviewed in Section 4.6.

4.6 RANK

This section gives you a chance to put together most of the ideas of the chapter in the same way that Section 2.3 collected the main ideas of the sections that preceded it.

KEY IDEAS

The Rank Theorem is the main result. By definition, rank A = dim Col A. But because rank A is also the dimension of Row A, the displayed equation in the theorem leads to the equation dim Row A + dim Nul $A = n$.

Equivalent Descriptions of Rank

The rank of an $m \times n$ matrix A may be described in several ways:

- the dimension of the column space of A, (our definition)
- the number of pivot positions in A, (from Theorem 6)
- the maximum number of linearly independent columns in A,
- the dimension of the row space of A, (from the Rank Theorem)
- the maximum number of linearly independent rows in A,
- the number of nonzero rows in an echelon form of A,
- the maximum number of columns in an invertible submatrix of A. (Supplementary Exercise 17 at the end of the chapter)

Pay attention to how Theorem 13 differs from the results in Section 4.3 about Col A: If you are interested in *rows* of A, use the nonzero rows of an echelon form B as a basis for Row A; if you are interested in the *columns* of A, only use B to obtain *information* about A (namely, to identify the pivot columns), and use the pivot columns of A as a basis for Col A. For Nul A, it is important to use the *reduced* echelon form of A.

When a matrix A is changed into a matrix B by one or more elementary row operations, the row space, null space, and column space of A may or may not be the same as the corresponding subspaces for B. The following table summarizes what can happen in this situation.

Effects of Elementary Row Operations

- Row operations do not affect the linear dependence relations among the columns. (That is, the columns of A have exactly the same linear dependence relations as the columns of any matrix that is row-equivalent to A.)
- Row operations usually change the column space.
- Row operations never change the row space.
- Row operations never change the null space.

The four subspaces shown in Figure 1 in the text are called the *fundamental subspaces* determined by A. (See Exercises 27–29.) The main difficulty here is to avoid confusion between Row A, Nul A, and Col A. The fourth subspace will appear again in Sections 6.1 and 7.4.

The following table lists all statements that are in the Invertible Matrix Theorem at this point in the course, arranged in the scheme used in Section 2.3 of this *Study Guide*. The statements in all three columns are equivalent when A is square ($m = n = p$). As before, a few extra statements have been added to make the table more symmetrical.

STATEMENTS FROM THE INVERTIBLE MATRIX THEOREM

Equivalent statements for an $m \times n$ matrix A.	Equivalent statements for an $n \times n$ square matrix A.	Equivalent statements for any $n \times p$ matrix A.
k. There is a matrix D such that $AD = I$.	a. A is an invertible matrix.	j. There is a matrix C such that $CA = I$.
*. A has a pivot position in every row.	c. A has n pivot positions.	*. A has a pivot position in every column.
h. The columns of A span $\mathbb{R}^m$.	m. The columns of A form a basis for $\mathbb{R}^n$.	e. The columns of A are linearly independent.
g. The equation $A\mathbf{x} = \mathbf{b}$ has at least one solution for each $\mathbf{b}$ in $\mathbb{R}^m$.	*. The equation $A\mathbf{x} = \mathbf{b}$ has a unique solution for each $\mathbf{b}$ in $\mathbb{R}^n$.	d. The equation $A\mathbf{x} = \mathbf{0}$ has only the trivial solution.
i. The transformation $\mathbf{x} \mapsto A\mathbf{x}$ maps $\mathbb{R}^n$ onto $\mathbb{R}^m$.	*. The transformation $\mathbf{x} \mapsto A\mathbf{x}$ is invertible.	f. The transformation $\mathbf{x} \mapsto A\mathbf{x}$ is one-to-one.
n. Col $A = \mathbb{R}^m$.	b. A is row equivalent to I.	q. Nul $A = \{\mathbf{0}\}$.
o. dim Col $A = m$.	l. A^T is invertible.	r. dim Nul $A = 0$.
*. rank $A = m$.	p. rank $A = n$.	*. rank $A = p$.